

## Questions on 8.2?

Answer the question below in your notes for review. This is NOT in your book.

Use sigma notation to rewrite each finite series. Then, calculate the given sum.

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32}; S_3$$
$$S_3 = \sum_{i=1}^3 a_i = a_1 + a_2 + a_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \boxed{\frac{7}{8}}$$
$$S_n = \frac{n(a_1 + a_n)}{2}$$

# I Am Having a Series Craving (For Some Math)!

## Geometric Series

# 8.3

pg.595-596 in your book.

**A geometric series is the sum of the terms of a geometric sequence.** Recall, that the sequence 1, 3, 9, 27, 81 is a geometric sequence because the ratio of any two consecutive terms is constant. Adding the terms creates the geometric series  $1 + 3 + 9 + 27 + 81$ .

Theresa raises her hand and claims that she has a "trick" for quickly calculating the sum of any geometric series. She asks members of the class to write any geometric series on the board. She boasts that she can quickly tell them how to determine the sum without adding all of the terms. Several examples are shown.

The constant ratio of this geometric sequence is 3 because  $\frac{3}{1} = \frac{9}{3} = \frac{27}{9} = \frac{81}{27} = 3$   
 Recall all geometric sequences have a constant ratio between successive terms.



<b>Paul:</b> "OK, so prove it! What is the sum of $1 + 3 + 9 + 27 + 81 + 243 + 729$ ?"	<b>Theresa:</b> "Multiply $729(3)$ and subtract 1. Then divide by 2."
<b>Stella:</b> "What is $5 + 20 + 80 + 320 + 1280 + 5120$ ?"	<b>Theresa:</b> "I will have the answer if I multiply $5120(4)$ , subtract 5, and then divide by 3."
<b>Julian:</b> "Let me see . . . How about $10 + 50 + 250 + 1250$ ?"	<b>Theresa:</b> "No problem. Multiply $1250(5)$ , subtract 10, and then divide by 4."
<b>Henry:</b> "Hmmm . . . I bet I can stump you with $10 + (-20) + 40 + (-80) + 160$ ."	<b>Theresa:</b> "Pretty sneaky with the negatives, Henry, but the method still works. Multiply $160(-2)$ and subtract 10. This time divide by $-3$ ."

1. Verify that Theresa is correct for each series.

yes

How can you tell all of the series are geometric?

2. What is Theresa's "trick"? Describe in words how to calculate the sum of any geometric sequence.

last term \* common ratio - first term  
common ratio - 1



pg.597 in your book

3. Use Theresa's "trick" to calculate  $1 + 2 + 4 + 8 + 16 + 32 + 64 + 128$ .

Show all work and explain your reasoning.

$$S_8 = \frac{128(2) - 1}{2 - 1} = \underline{\underline{255}}$$

$$\begin{aligned} r &= 2 \\ g_1 &= 1 \\ g_n &= 128 \end{aligned}$$

Remember,  $g_n = g_1 r^{n-1}$ .



Theresa's "trick" really isn't a trick. It is known as Euclid's Method. An example of this method, along with a justification for each step, is shown.

Compute  $\sum_{i=1}^5 3^{i-1}$ .

$S_5 = 1 + 3 + 9 + 27 + 81$  • The common ratio is 3.

$3S_5 = 3 + 9 + 27 + 81 + 243$  • Write  $3S_n$  above the original series. Multiply each term of the original series by the common ratio. Line up each product above the original series.

$S_5 = 1 + 3 + 9 + 27 + 81$

$3S_5 = 3 + 9 + 27 + 81 + 243$  • Subtract to determine  $3S_n - S_n = 2S_n$ .

$-S_5 = -(1 + 3 + 9 + 27 + 81)$

$2S_5 = -1 + 243$

$\frac{2S_5}{2} = \frac{242}{2}$  • Divide by 2.

$S_5 = 121$

In all of the examples, Theresa knew that she could calculate each sum by first multiplying the last term by the common ratio and subtracting the first term. Then she could divide that quantity by one less than the common ratio.

In other words,  $S_n = \frac{(\text{Last Term})(\text{Common Ratio}) - (\text{First Term})}{(\text{Common Ratio} - 1)}$ .

4. Analyze the worked example.
- a. In the worked example, why multiply both sides of the equation by 3? Does the algorithm still work if you multiply by a different number? Explain your reasoning.

pg.598 in your book

pg.598 in your book

b. Why do you always divide by one less than the common ratio?

The formula to compute any geometric series becomes  $S_n = \frac{g_n(r) - g_1}{r - 1}$  where  $g_n$  is the last term,  $r$  is the common ratio, and  $g_1$  is the first term.

$$S_n = \frac{g_n(r) - g_1}{r - 1}$$

Formula #1

5. Apply Euclid's Method to compute each.  $r=10$ a.  $1 + 10 + 100 + \dots + 1,000,000$ 

$$S_n = \frac{1000000(10) - 1}{10 - 1} = 1,111,111$$

b.  $10 + 20 + 40 + 80 + 160 + 320$ 

$$S_6 = \frac{320(2) - 10}{2 - 1} = 630$$

c.  $\sum_{k=1}^8 5^{k-1} = 5^{1-1} + 5^{2-1} + \dots + 5^{8-1}$   $r=5$ 

$$S_8 = \frac{5^8(5) - 1}{5 - 1} = 97,656$$

Do you need to know all of the terms? How can you determine just the terms that you need? Remember to work efficiently, looking for patterns and applying formulas that you already know.

d. A sequence with 9 terms, a common ratio of 2, and a first term of 3.  $r=2$ 

$$S_9 = \frac{768(2) - 3}{2 - 1} = 1533$$

$$\left\{ \begin{array}{l} g_1 = 3 \\ g_a = 3(2^{a-1}) \\ g_a = 768 \end{array} \right.$$

## pg.598 in your book

Recall previously you used long division to determine each quotient:

**Polynomial Long Division**

**Rewritten Using the Reflexive and Commutative Properties of Equality**

**Example 1**

$$\frac{r^3 - 1}{r - 1} = r^2 + r + 1$$



$$1 + r + r^2 = \frac{r^3 - 1}{r - 1}$$

**Example 2**

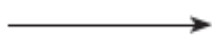
$$\frac{r^4 - 1}{r - 1} = r^3 + r^2 + r + 1$$



$$1 + r + r^2 + r^3 = \frac{r^4 - 1}{r - 1}$$

**Example 3**

$$\frac{r^5 - 1}{r - 1} = r^4 + r^3 + r^2 + r + 1$$



$$1 + r + r^2 + r^3 + r^4 = \frac{r^5 - 1}{r - 1}$$

## pg.599 in your book

Each Example represents a geometric series, where  $r$  is the common ratio and  $g_1 = 1$ . Each geometric series can be written in summation notation.

**Example 1:**  $n = 3$       $\sum_{i=1}^3 r^{i-1}$      or      $\sum_{i=0}^2 r^i$

**Example 2:**  $n = 4$       $\sum_{i=1}^4 r^{i-1}$      or      $\sum_{i=0}^3 r^i$

- For each Example, explain why the power of the common ratio in the summation notation is different, yet still represents the series.
- Identify the number of terms in the series in Example 3, and then write the series in summation notation.
- Use the pattern generated from repeated polynomial long division to write a formula to compute any geometric series  $1 + r + r^2 + r^3 + \cdots + r^{n-1}$  where  $n$  is the number of terms in the series,  $r$  is the common ratio, and  $g_1 = 1$ .

$$\sum_{i=0}^n r^i = \underline{\hspace{2cm}}$$

pg.599 in your book

You can show a proof of  $S_n = \frac{r^n - 1}{r - 1}$  where  $S_n$  is a series in the form  $r^0 + r^1 + r^2 + r^3 + \dots + r^{n-1}$  with  $n$ -terms and a common ratio  $r$ .

$$S_n = r^0 + r^1 + r^2 + r^3 + \dots + r^{n-1}$$

$$rS_n = r^1 + r^2 + r^3 + \dots + r^{n-1} + r^n$$

$$S_n = r^0 + r^1 + r^2 + \dots + r^{n-2} + r^{n-1}$$

- Write  $rS_n$  above the original series.
- Multiply each term by  $r$ . Line up each product above the original series.

$$rS_n = r^1 + r^2 + r^3 + \dots + r^{n-1} + r^n$$

$$- S_n = -(1 + r^1 + r^2 + r^3 + \dots + r^{n-1})$$

$$rS_n - S_n = -1 + r^n$$

$$S_n(r - 1) = r^n - 1$$

$$\frac{S_n(r - 1)}{(r - 1)} = \frac{(r^n - 1)}{(r - 1)}$$

- Subtract  $rS_n - S_n$ .
- Eliminate terms that subtract to 0.

- Divide by  $(r - 1)$ .

$$S_n = \frac{r^n - 1}{r - 1}$$

Formula #2  
\* Use when given  $r \neq 1$

pg.600 in your book

4. Identify the number of terms, the common ratio, and  $g_1$  for each series. Then compute each.

a.  $1 + 2^1 + 2^2 + 2^3 + 2^4$   $r=2$   
 $n=5$

Notice that  $g_1 = 1$   
in each series.

b.  $1 + 5 + 25 + 125 + 625$

$r=5$   
 $n=5$



you could  
use

$$S_n = \frac{r^n - 1}{r - 1}$$

c.  $1 + (-2) + 4 + (-8) + 16 + (-32)$

$r=-2$   
 $n=6$

pg.601 in your book

Formula #3: use when given first term, # of terms (n) & ratio.

The formula to compute a geometric series that Perry used is

$$S_n = \frac{g_1(r^n - 1)}{r - 1}$$

Recall Euclid's Method to compute a geometric series is

$$S_n = \frac{g_n(r) - g_1}{r - 1}$$

Formula #1

Use when given last term, first term, and ratio

pg.602 in your book

7. Rewrite each series using summation notation.

a.  $4 + 12 + 36 + 108 + 324$

b.  $64 + 32 + 16 + 8 + 4 + 2 + 1$

8. Compute each geometric series.

a.  $\sum_{i=1}^4 6^{i-1}$

b.  $10 \sum_{i=0}^4 3^i$

c.  $6 \sum_{i=0}^4 \left(\frac{1}{3}\right)^i$





## NOT in your book

2. Ryan and Morgan have competed in the Boston Marathon for 9 consecutive years. In Ryan's 1st year, he ran the marathon in 3.5 hours and he has steadily decreased his time by 3% each year. In Morgan's 1st year, he ran the marathon in 3.3 hours and he has steadily decreased his time by 2% each year.
- a. Which of the 2 runners had the fastest time in their 9th marathon? Round decimals to the nearest hundredth.
- b. Which of the 2 runners had the fastest total time if they combine each of their 9 marathon times? Round decimals to the nearest hundredth.

Homework  
Finish Lesson 8.3