Questions on 7-6 Worksheet?

FIRST PROBLEM

2010 is $t=0$

$$\frac{dW}{dt} = \frac{1}{25}(W-300)$$

a) $$\left.\frac{dW}{dt}\right]_{t=0} = \frac{1}{25}(W(0)-300)$$

$$= \frac{1}{25}(1400-300)$$

$$= 44$$

$(0,1400) \rightarrow$ $y = 1400 + 44t$

$W(t) = 1400 + 44t$

$\Rightarrow W\left(\frac{1}{4}\right) = 1400 + 44\left(\frac{1}{4}\right)$

$W\left(\frac{1}{4}\right) = 1411$ tons

b) $$\frac{dW}{dt} = \frac{1}{25}(W-300) = \frac{1}{25}W - 12$$

$$\frac{d^2W}{dt^2} = \frac{1}{25}\cdot\frac{dW}{dt}$$

$$\frac{d^2W}{dt^2} = \frac{1}{25}\left(\frac{1}{25}(W-300)\right)$$

$$\frac{d^2W}{dt^2} = \frac{1}{625}(W-300)$$

$W \geq 1400$ tons (increasing)

$\frac{d^2W}{dt^2} > 0$ from $0 \leq t \leq \frac{1}{4}$

So (a) is an underestimate

c) $W = 1100e^{t/25} + 300, \quad 0 \leq t \leq 20$

7.4 Exponential Growth and Decay & 7.5 Logistic Growth

Direct and Inverse Proportion

$y = kx$

$y = \frac{k}{x}$

The Law of Exponential Change

If y changes at a rate proportional to the amount present (that is, if $\frac{dy}{dt} = ky$), and if $y = y_o$ when $t = 0$, then

$$y = y_o e^{kt}$$

The constant k is the **growth constant** if $k > 0$ or the **decay constant** if $k < 0$.

Examples

1. The rate of change of the volume, V, of water in a tank with respect to time t is directly proportional to the square root of the volume. Which of the following is a differential equation that describes this relationship? (No calculator)

~~(A)~~ $V(t) = k\sqrt{t}$

~~(B)~~ $V(t) = k\sqrt{V}$

(C) $\frac{dV}{dt} = k\sqrt{t}$

~~(D)~~ $\frac{dV}{dt} = \frac{k}{\sqrt{V}}$

(E) $\frac{dV}{dt} = k\sqrt{V}$

2. A puppy weighs 2.0 pounds at birth and 3.5 pounds two months later. If the weight of the puppy during its first 6 months is increasing at a rate proportional to its weight, then how much will the puppy weigh when it is 3 months old? (Calculator allowed)

(A) 4.2 pounds (B) 4.6 pounds (C) 4.8 pounds (D) 5.6 pounds (E) 6.5 pounds

$y = y_0 e^{kt}$

when $t=0$, $y=2 \rightarrow y_0$

$t=2$, $y=3.5$

$t=3$, $y=?$

$\frac{3.5}{2} = \frac{2e^{k\cdot 2}}{2}$

$\ln 1.75 = \cancel{\ln} e^{2k}$

$\frac{\ln 1.75}{2} = \frac{2k}{2}$

$\boxed{\frac{\ln 1.75}{2} = k}$

$\boxed{0.279 = k}$

$y = 2e^{0.279(3)}$

$y = 4.61886$

Logistic Growth

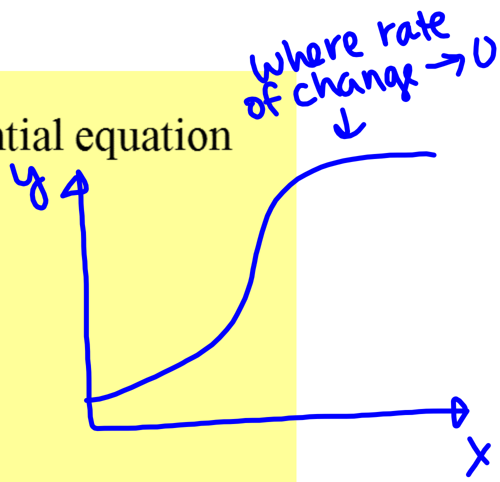
The solution of the general logistic differential equation

$$\frac{dP}{dt} = kP(M-P)$$

is

$$P = \frac{M}{1 + Ae^{-(Mk)t}}$$

where A is a constant determined by an appropriate initial condition. The **carrying capacity** M and the **growth constant** k are positive constants.



3. A rumor spreads among a population of N people at a rate proportional to the number p of people who have heard the rumor and the number of people who have not heard the rumor. Which of the following differential equations could be used to model this situation, if t is time and k is a positive constant? (No calculator)

~~(A)~~ $\frac{dp}{dt} = kp$

(B) $\frac{dp}{dt} = kp(N-p)$

(C) $\frac{dp}{dt} = kp(p-N)$

~~(D)~~ $\frac{dp}{dt} = kt(N-t)$

~~(E)~~ $\frac{dp}{dt} = kp(t-N)$

p = # of people who have heard rumor

N - p = # of people who have NOT heard rumor

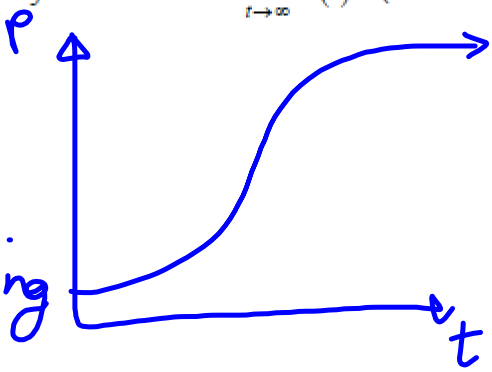
Example

4. The population, $P(t)$, of a species satisfies the logistical differential equation

$$\frac{dP}{dt} = P\left(2 - \frac{P}{5000}\right)$$

where the initial population $P(0) = 3000$ and t is the time in years. What is $\lim_{t\to\infty} P(t)$? (No calculator)

$$\lim_{t\to\infty} P(t) = 0$$



We are reaching our carrying capacity as $t \to \infty$ and our rate of change $\to 0$

Feb 10-8:42 AM

Half-life

The **half-life** of a radioactive substance with rate constant k $(k > 0)$ is

$$\text{half-life} = \frac{\ln 2}{k}.$$

Compound Interest Formulas

If the interest is added continuously at a rate proportional to the amount in the account, you can model the growth of the account with the initial value problem:

Differential equation: $\frac{dA}{dt} = rA$

Initial condition: $A(0) = A_o$

The amount of money in the account after t years at an annual interest rate r:

$A(t) = A_o e^{rt}$.

$A = Pe^{rt}$

Example

Suppose you deposit \$500 in an account that pays 5.3% annual interest. How much will you have 4 years later if the interest is (a) compounded continuously? (b) compounded monthly?

Answer

Let $A_o = 500$ and $r = 0.053$.

a. $A(4) = 500e^{(0.053)(4)} = 618.07$

b. $A(4) = 500\left(1 + \dfrac{0.053}{12}\right)^{(12)(4)} = 617.79$

Newton's Law of Cooling

The rate at which an object's temerature is changing at any given time is roughly proportional to the difference between its temperature and the temperature of the surrounding medium. If T is the temperature of the object at time t, and T_S is the surrounding temperature, then

$$\frac{dT}{dt} = -k\left(T - T_S\right). \qquad (1)$$

Since $dT = d(T - T_S)$, rewrite (1)

$$\frac{d}{dt}\left(T - T_S\right) = -k(T - T_S)$$

Its solution, by the law of exponential change, is

$$T - T_S = \left(T_O - T_S\right)e^{-kt},$$

Where T_O is the temperature at time $t = 0$.

Example

A temperature probe is removed from a cup of coffee and placed in water that has a temperature of $T_s = 4.5°C$.
Temperature readings T, as recorded in the table below, are taken after 2 sec, 5 sec, and every 5 sec thereafter.

Estimate
(a) the coffee's temperature at the time the temperature probe was removed.
(b) the time when the temperature probe reading will be 8°C.

Table 6.1 Experimental Data

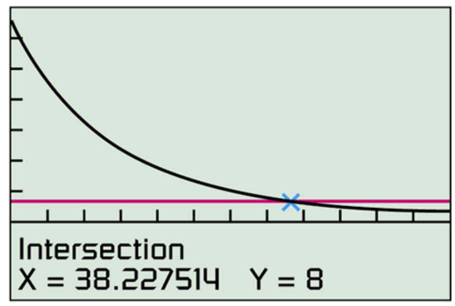
Time (sec)	T (°C)	$T - T_s$ (°C)
2	64.8	60.3
5	49.0	44.5
10	31.4	26.9
15	22.0	17.5
20	16.5	12.0
25	14.2	9.7
30	12.0	7.5

Answer

According to Newton's Law of Cooling, $T - T_s = (T_O - T_s)e^{kt-}$,
where $T_s = 4.5$ and T_O is the temperature of the coffee at $t = 0$.
Use exponential regression to find that $T - 45 = 61.55(0.9277)^t$
is a model for the $(t,\ T - T_s) = t, T - 4.5)$ data. Thus
$T = 4.5 + 61.66(0.9277)^t$ is a model of the (t, T), data.

(a) At time $t = 0$ the temperature was
$T = 4.5 + 61.66(0.9277) \approx 66.16°C$

(b) The figure below shows the graphs of $y = 8$
and $y = T = 4.5 + 61.66(0.9277)^t$



[0, 60] by [–20, 70]
(b)

5. A hard-boiled egg with a temperature of 98°C (T_0) is put into 18°C (T_s) water. After 5 minutes (t), the egg's temperature is 38° (T when $t=5$). How long from the time the egg is put into the water will it take the egg's temperature to reach 20°C?

$$T - T_s = (T_0 - T_s)e^{-kt}$$

$$T - 18 = (98-18)e^{-kt}$$

$$T - 18 = 80e^{-kt}$$

$$T = 18 + 80e^{-kt}$$

$t=5$

$$38 = 18 + 80e^{-k(5)}$$

$$\underset{-18}{38} = 80e^{-5k}$$

$$\frac{20}{80} = \frac{80e^{-5k}}{80}$$

$$\ln 1/4 = \ln e^{-5k}$$

$$\frac{\ln 1/4}{-5} = \frac{-5k}{-5}$$

$$\boxed{\frac{\ln 1/4}{-5} = k}$$

$$20 = 18 + 80e^{+\left(\frac{\ln 1/4}{-5}\right)t}$$

$$\frac{2}{80} = \frac{80e^{\left(\frac{\ln 1/4}{5}\right)t}}{80}$$

$$\ln\frac{1}{40} = \ln e^{\left(\frac{\ln 1/4}{5}\right)t}$$

$$\frac{\ln 1/40}{\frac{\ln 1/4}{5}} = \frac{\frac{\ln 1/4}{5} \cdot t}{\frac{\ln 1/4}{5}}$$

$$13.3 = t$$

mins

$$y = y_0 e^{kt}$$

6. (Calculator allowed) Population y grows according to the equation $\frac{dy}{dt} = ky$ where k is a constant, and t is measured in years. If the population doubles every 10 years then the value of k is

(A) 0.069 (B) 0.200 (C) 0.301 (D) 3.322 (E) 5.000

Homework

7.4 pg.361-2 #3-27(X3);

7.5 pg.373 #~~3-33(X3)~~ #23-26