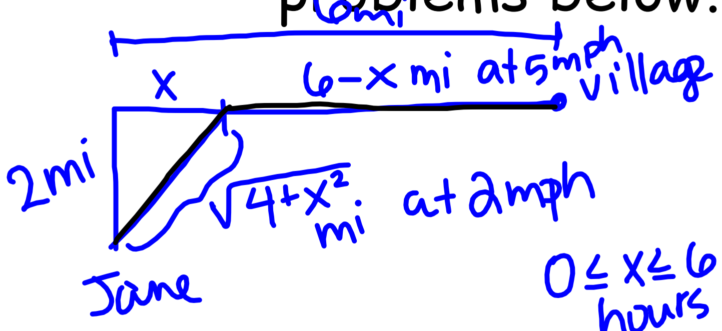
Find two numbers whose sum is 20 and whose product is as large as possible.

You have been asked to design a one-liter oil can shaped like a right circular cylinder. What dimensions will use the least material?

Questions on HW? If not, work on the problems below...



total amt. of time

$$f(x)=\frac{\sqrt{4+x^2}}{2}+\frac{6-x}{5}$$

$0 \le x \le 6$
hours

$$f'(x)=\frac{1}{2}\cdot\frac{1}{2\sqrt{4+x^2}}(2x)-\frac{1}{5}=\frac{x}{2\sqrt{4+x^2}}-\frac{1}{5}$$

$$0=\frac{x}{2\sqrt{4+x^2}}-\frac{1}{5}$$

$$\frac{1}{5}=\frac{x}{2\sqrt{4+x^2}}$$

$$5x=2\sqrt{4+x^2}$$

$$25x^2=4(4+x^2)$$

$$25x^2=16+4x^2$$

$$\frac{\cancel{21}x^2}{\cancel{21}}=\frac{16}{21}$$

$$\sqrt{x^2}=\sqrt{\frac{16}{21}}$$

$$x=\pm\frac{4}{\sqrt{21}}$$

$$x=\frac{4}{\sqrt{21}}$$

endpoints

$f(0)=2.2$

$f(6)=3.16$

$\star\ f\left(\frac{4}{\sqrt{21}}\right)=2.12$

Jane should land her boat $\frac{4}{\sqrt{21}}\approx 0.87$ mi. down the shoreline from the point nearest her boat.

Nov 3-2:38 PM

5.5 Linearization and Differentials

Linearization or Linear Approximation

$y - y_1 = m(x - x_1)$

If f is differentiable at $\underline{x = a}_{x_1}$, then the equation of the tangent line, $L(x) = f(\underset{x_1}{a}) + f'(a)(x - \underset{x_1}{a})$, defines the **linearization of f at a**. The approximation $f(x) \approx L(x)$ is the **standard linear approximation of f at a**. The point $x = a$ is the **center** of the approximation.

$(a, f(a))$

$f'(a)$

Example

Find the linearization of $f(x) = \cos x$ at $x = \pi/2$ and use it to approximate $\cos 1.75$ without a calculator.

$f(x) = \cos(\pi/2) = 0$

$a = \frac{\pi}{2}$ $(\pi/2, 0)$

$L(x) = 0 + -1\left(x - \frac{\pi}{2}\right)$

$L(x) = -x + \pi/2$

$\cos 1.75 = -1.75 + \pi/2$

$f'(x) = -\sin x$

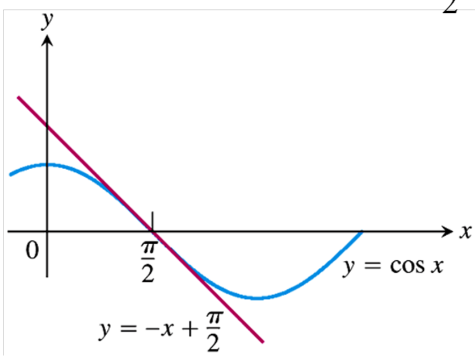
$f\left(\frac{\pi}{2}\right) = -\sin(\pi/2)$

$f(\pi/2) = -1$

Answer

Since $f(\pi/2) = \cos(\pi/2) = 0$, the point of tangency is $(\pi/2, 0)$. The slope of the tangent line is $f'(\pi/2) = -\sin(\pi/2) = -1$. Thus $L(x) = 0 + (-1)\left(x - \frac{\pi}{2}\right) = -x + \frac{\pi}{2}$.

To approximate $\cos 1.75 = f(1.75) \approx L(1.75) = -1.75 + \frac{\pi}{2}$.



Differentials

Let $y = f(x)$ be a differentiable function. The **differential** dx is an independent variable. The **differential** dy is $dy = f'(x)dx$.

Example

Find the differential dy and evaluate dy for the given value of x and dx

$y = x^5 + 2x,\ x = 1,\ dx = 0.01$

$f'(x) = 5x^4 + 2$

$dy = (5x^4 + 2)dx$

$dy = (5 \cdot 1^4 + 2)(0.01)$

$dy = 7(0.01)$

$dy = 0.07$

Answer

$$dy = \left(5x^4 + 2\right)dx$$

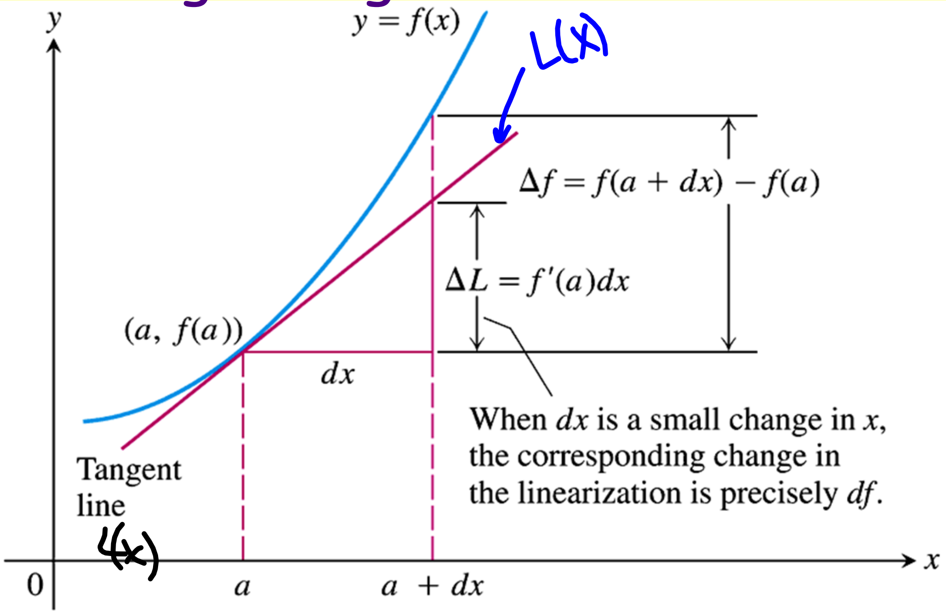
$$dy = (5+2)(0.01)$$

$$= 0.07$$

Differential Estimate of Change

Let $f(x)$ be differentiable at $x = a$. The approximate change in the value of f when x changes from a to $a + dx$ is $df = f'(a)dx$.

Estimating Change with Differentials



$dy = f'(x)\,dx$
$df = f'(a)\,dx$

Example

The radius of a circle increases from $a = 5$ m to 5.1 m. Use dA to estimate the increase in the circle's area A. $3r^2$

$d\cancel{x} = 5.1 - 5 = 0.1$

$A = \pi r^2$

$dA = 2\pi r\,dr$ m^2

$dA = 2\pi(5)(0.1)$ m^2

$dA = \pi\,m^2$

Answer

Since $A = \pi r^2$, the estimated increase is

$$dA = 2\pi r\,dr$$

$$= 2\pi(5)(0.1)$$

$$= \pi \text{ m}^2$$

Examples

1. Use linearization to approximate the value of $f(1.2)$ if $\frac{d}{dx}(f(x)) = -\frac{x}{y}$ and $f(x)$ passes through the point $(2, 1)$.

$f(2)=1$

$f'(2)= \frac{-2}{1} = -2$

$L(x)= 1-2(x-2)$

$L(x) = 1-2x+4$

$L(x)=-2x+5$

$L(1.2) = -2(1.2)+5$

$L(1.2) = 2.6$

2. Find the linearization of $f(x) = \cos x$ at $x = \frac{\pi}{2}$. Use it to estimate $f\left(\frac{9\pi}{16}\right)$.

From before... $L(x)=x+\pi/2$

$L\left(\frac{9\pi}{16}\right) = -\frac{9\pi}{16} + \frac{\pi}{2} = -\pi/16$

3. If $f'(x, y) = \frac{-4x-2y}{x+2y}$ and $f(-1) = 2$, use linear approximation to estimate $f(-0.9)$.

Examples - AP Problems

1. Let f be a differentiable function, such that $f(3) = 2$ and $f'(3) = 5$. If the tangent line to the graph of f is used to find an approximation to a zero of f, that approximation is

(A) 0.4 (B) 0.5 (C) 2.6 (D) 3.4 (E) 5.5

2.

x	-1.5	-1.0	-0.5	0	0.5	1.0	1.5
$f(x)$	-1	-4	-6	-7	-6	-4	-1
$f'(x)$	-7	-5	-3	0	3	5	7

(a) Write the equation of the tangent to the graph of f at the point where $x = 1$. Use this line to approximate the value of $f(1.2)$.

(b) Is the approximation greater or less than $f(1.2)$? How do you know?

5. Let f be a function with $f(1) = 4$ such that for all points (x, y) on the graph of f the slope is given by $\dfrac{3x^2+1}{2y}$.

(a) Find the slope of the graph of f at the point where $x = 1$.

(b) Write an equation for the line tangent to the graph of f at $x = 1$ and use it to approximate $f(1.2)$.

Homework

5.5 pgs.246-247 #3-42 (X3)