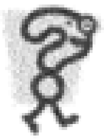
4.1 The Chain Rule

Review...

What is a *composite function*?



- A combination of two functions, like $f(x)$ and $g(x)$
- One function replaces all of the xes in the other function
- Notation: $(f \circ g)(x)$ or $(f(g(x))$
- Example:

  If $f(x) = x^2$ and $g(x) = 2x + 1$, then

  $f(g(x)) = (2x + 1)^2$

  $g(f(x)) = 2x^2 + 1$

Nov 3-2:38 PM

Practice!
Write the composite of each of the following functions

1. $f(x) = x^3$, $g(x) = \sin x$ Find $(f \circ g)(x)$

$(\sin x)^3$ or $\sin^3 x$

2. $r(\theta) = 2\theta$, $s(\theta) = 3\theta + 5$ Find $(r \circ s)(\theta)$

$2(3\theta+5)$

3. $y(t) = \cos t$, $x(t) = t^2 - 1$ Find $(y \circ x)(t)$

$\cos(t^2-1)$

Here are the answers.

1. $(f \circ g)(x) = (\sin x)^3 = \sin^3 x$

2. $(r \circ s)(\theta) = 2(3\theta + 5)$

3. $(y \circ x)(t) = \cos(t^2 - 1)$

How did you do?? Do you think you understand composite functions?

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Another way to look at it . . . .

You can think of the two functions in a composite function as an "outside" function and an "inside" function.

For example, look at the composite function

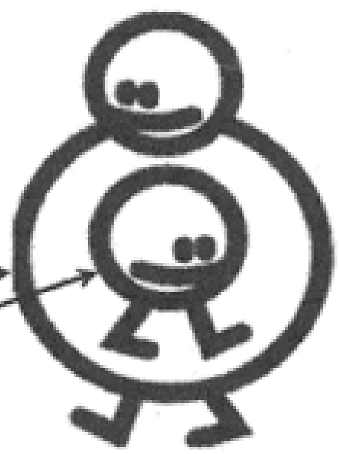
$$y = \cos^2 x$$

Another way to write this function is

$$y = (\cos x)^2$$

The "outside" function is $y = u^2$

The "inside" function is $u = \cos x$



Name the outside and inside functions in each of the following composite functions.

Function	Outside	Inside
$y = \sin(x^2 + 3)$	$\sin u$	$u = x^2+3$
$y = (3x - 2)^3$	u^3	$u = 3x-2$
$y = \cos(x^2 + x)$	$\cos u$	$u = x^2+x$
$y = \sec(\tan x)$	$\sec u$	$u = \tan x$
$y = 4\sqrt{x^2 - 3}$	$4\sqrt{u}$	$u = x^2-3$

Answers

Name the outside and inside functions in each of the following composite functions.

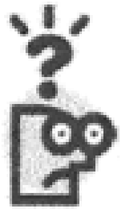
Function	Outside	Inside
$y = \sin(x^2 + 3)$	$y = \sin u$	$u = x^2 + 3$
$y = (3x - 2)^3$	$y = u^3$	$u = 3x - 2$
$y = \cos(x^2 + x)$	$y = \cos u$	$u = x^2 + x$
$y = \sec(\tan x)$	$y = \sec u$	$u = \tan x$
$y = 4\sqrt{x^2 - 3}$	$y = 4\sqrt{u}$	$u = x^2 - 3$

Now let's get to the Chain Rule . . .

It says . . .

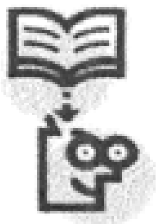
If f is differentiable at the point $u = g(x)$, and g is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$



In plain English, the Chain Rule says . . .

If you have a composite function $(f \circ g)(x)$, you take the derivative of this function by taking the derivative of the outside function (leaving the inside function alone), and multiply by the derivative of the inside function



Here's an example . . .

$$f(x) = (3x - 2)^2$$

$$f'(x) = 2(3x-2)^{2-1} \cdot \frac{d}{dx}(3x-2)$$

$$f'(x) = 2(3x-2) \cdot 3$$

$$f'(x) = 6(3x-2) \text{ or } 18x-12$$

Now find the derivatives of the following . . .

$u = x^2+3$
$u' = 2x$

$y = \sin(x^2+3)$ $\quad y' = \cos(x^2+3) \cdot 2x = 2x\cos(x^2+3)$

$y = (3x-2)^3$ $\quad y' = 3(3x-2)^2 \cdot 3 = 9(3x-2)^2$

$y = \cos(x^2+x)$ $\quad y' = -\sin(x^2+x)\cdot(2x+1) =$
$-(2x+1)\sin(x^2+x)$

$y = \sec(\tan x)$ $\quad y' = \sec(\tan x)\cdot\tan(\tan x)\cdot\sec^2 x$

$y = 4\sqrt{x^2-3x} = 4(x^2-3x)^{1/2}$

$y' = 4\cdot\frac{1}{2}(x^2-3x)^{-1/2}\cdot(2x-3) =$
$2(x^2-3x)^{-1/2}(2x-3) = \dfrac{2(2x-3)}{\sqrt{x^2-3x}}$

Here are the answers!

$y' = \cos(x^2+3)(2x) = 2x\cos(x^2+3)$

$y' = 3(3x-2)^2(3) = 9(3x-2)^2$

$y' = -\sin(x^2+x)(2x+1)$

$y' = \sec(\tan x)\tan(\tan x)(\sec^2 x)$

$y' = 2(x^2-3x)^{-1/2}(2x-3) = \dfrac{2(2x-3)}{\sqrt{x^2-3x}}$

EXAMPLES

1. If $y = \sin^2(3x-2)$ find y'.

$y = (\sin(3x-2))^2$

outside: u^2

inside: $\sin(3x-2)$

inside$(3x-2)$

$y' = 2\cdot(\sin(3x-2))\cdot\cos(3x-2)\cdot 3$

$= 6\sin(3x-2)\cdot\cos(3x-2)$

$2\sin a\cos a = \sin 2a$

$= 3\sin[2(3x-2)]$

$= 3\sin(6x-4)$

2. $\dfrac{d}{dx}\left((1+\cos(2x))^2\right) = 2(1+\cos(2x))\cdot -\sin(2x)\cdot 2$

$= -4(1+\cos(2x))(\sin 2x)$

u^2

$1+\cos(2x)$

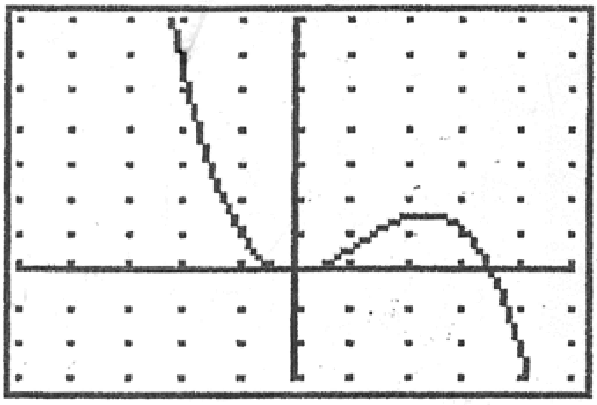
$\dfrac{d}{dx} = -\sin(2x)\cdot 2$

3. Write the equation of the line tangent to $f(x) = \sqrt{3x^2-2}$ at the point where $x = 3$.

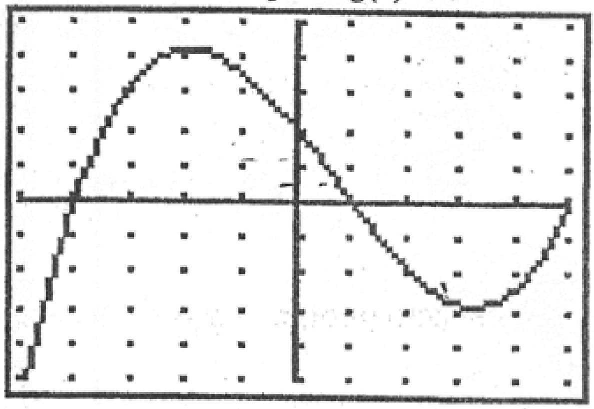
The Chain Rule graphically:

Let f and g be the functions defined below

Graph of f(x)



Graph of g(x)



Let h(x) = f(g(x))

a) Evaluate h(-2), h(1), and h(2)

b) Is h′(-1) positive, negative, or equal to zero. Justify your answer.

c) Estimate the sign of h′(-2), h′(1), and h′(2).

Homework

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