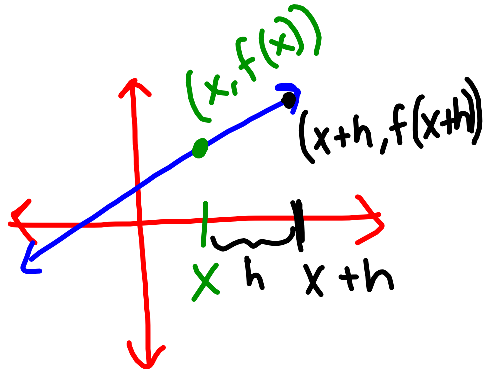
2.1 Rates of Change and Limits

Average Rate of Change:

- slope
- can find on curves

$$\frac{\Delta y}{\Delta x}$$

$$\frac{f(x+h) - f(x)}{(x+h) - x}$$



An object is dropped from rest from the top of a tall building falls $y = 16t^2$ feet in the first t seconds.

*What is the average speed during the first 4 seconds of fall?

(0, 0)
(4, 256)

$$\frac{256 - 0}{4 - 0} = \frac{256}{4} = 64$$

ft/sec

Definition of a Limit:

Let c and L be real numbers. The function f has limit L as x approaches c if, given any positive number ε, there is a positive number δ such that for all x,

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

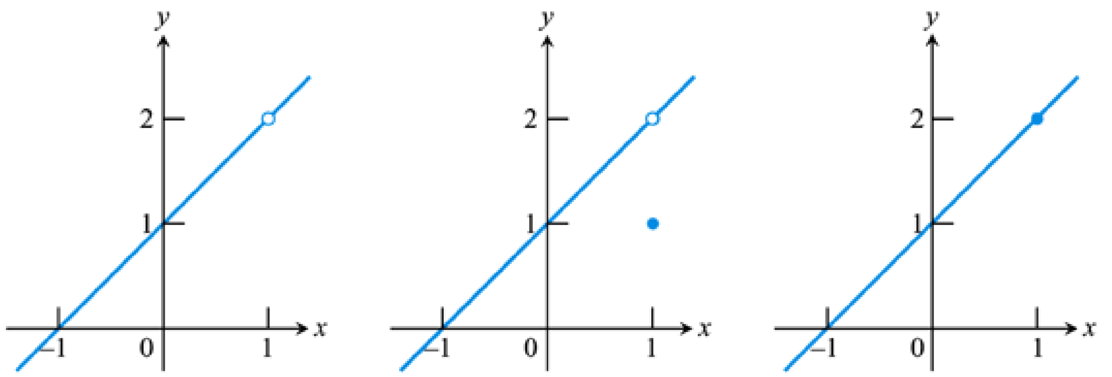
We write

$$\lim_{x \to c} f(x) = L$$

The sentence $\lim_{x \to c} f(x) = L$ is read, "The limit of f of x as x approaches c equals L". The notation means that the values $f(x)$ of the function f approach or equal L as the values of x approach (but do not equal) c.

Examples:

The function f has limit 2 as $x \to 1$ even though f is not defined at 1.
The function g has limit 2 as $x \to 1$ even though $g(1) \neq 2$.
The function h is the only one whose limit as $x \to 1$ equals its value at $x = 1$.



(a) $f(x) = \dfrac{x^2 - 1}{x - 1}$ (b) $g(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$ (c) $h(x) = x + 1$

Theorem 1: Properties of Limits- pg.61-62

If L, M, c, and k are real numbers and

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M, \text{ then}$$

1. *Sum Rule*: $\lim_{x\to c}\left(f(x)+g(x)\right) = L+M$

   The limit of the sum of two functions is the sum of their limits.

2. *DifferenceRule*: $\lim_{x\to c}\left(f(x)-g(x)\right) = L-M$

   The limit of the difference of two functions is the difference of their limits.

3. *Product Rule*: $\lim_{x\to c}\left(f(x)\bullet g(x)\right) = L\bullet M$

   The limit of the product of two functions is the product of their limits.

4. *Constant Multiple Rule*: $\lim_{x\to c}\left(k\bullet f(x)\right) = k\bullet L$

   The limit of a constant times a function is the constant times the limit of the function.

5. *Quotient Rule*: $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{L}{M},\ M \neq 0$

   The limit of the quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.

6. *Power Rule*: If r and s are integers, $s \neq 0$, then

   $$\lim_{x\to c}\left(f(x)\right)^{\frac{r}{s}} = L^{\frac{r}{s}}$$

   provided that $L^{\frac{r}{s}}$ is a real number.
   The limit of a rational power of a function is that power of the limit of the function, provided the latter is a real number.

Other properties of limits:

$\lim_{x\to c}(k) = k$ k is a constant #

$\lim_{x\to c}(x) = c$

You try...

Use any of the properties of limits to find

$$\lim_{x \to c}\left(3x^3 + 2x - 9\right) = \lim_{x \to c} 3x^3 + \lim_{x \to c} 2x - \lim_{x \to c} 9$$

$$= 3c^3 + 2c - 9$$

Answer:

$$\lim_{x \to c}\left(3x^3 + 2x - 9\right) = \lim_{x \to c} 3x^3 + \lim_{x \to c} 2x - \lim_{x \to c} 9 \qquad \text{sum and difference rules}$$

$$= 3c^3 + 2c - 9 \qquad \text{product and multiple rules}$$

Sep 12-2:26 PM

Theorem 2-Polynomial and Rational Functions-

1. If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$ is any polynomial function and c is any real number, then

$$\lim_{x \to c} f(x) = f(c) = a_n c^n + a_{n-1}c^{n-1} + \ldots + a_0$$

2. If $f(x)$ and $g(x)$ are polynomials and c is any real number, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)}, \text{ provided that } g(c) \neq 0.$$

You try...

Use Theorem 2 to find $\lim_{x \to 5} \underbrace{(4x^2 - 2x + 6)}_{}$ $f(x)$

$\lim_{x \to 5} (4x^2 - 2x + 6) = f(5) = 4(5)^2 - 2(5) + 6 = \underline{96}$

$$\lim_{x \to 5} (4x^2 - 2x + 6) = 4(5)^2 - 2(5) + 6$$

$$= 4(25) - 10 + 6$$

$$= 100 - 10 + 6$$

$$= 96$$

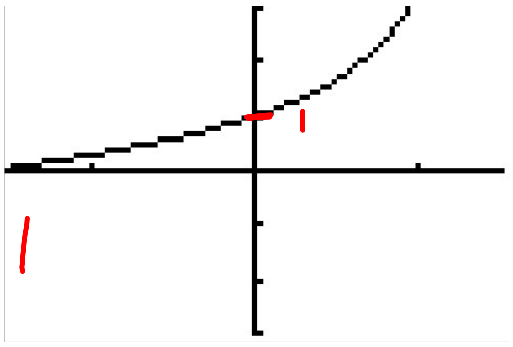
Sep 13-2:16 PM

We can also find limits by substitution with many other types of functions.

You try...

Find $\lim_{x \to 0} \frac{1+\sin x}{\cos x}$ $= f(0) = \frac{1+\sin 0}{\cos 0}$

$= \frac{1+0}{1} = 1$



Solve graphically:

The graph of $f(x) = \frac{1+\sin x}{\cos x}$ suggests that the limit exists and is 1.

Confirm Analytically:

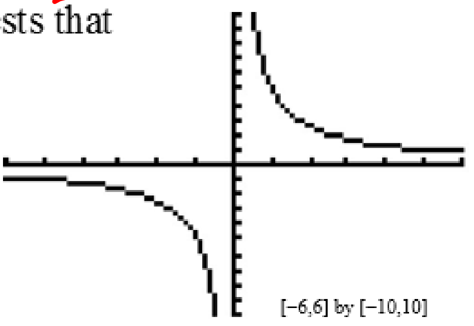
Find $\lim_{x \to 0} \frac{1+\sin x}{\cos x} = \frac{\lim_{x \to 0}(1+\sin x)}{\lim_{x \to 0} \cos x} = \frac{(1+\sin 0)}{\cos 0}$

$= \frac{1+0}{1} = 1$

Here's another one...

Find $\lim_{x\to 0} \frac{5}{x}$ $= f(0) = \frac{5}{0}$ ... ? look at graph...
DNE

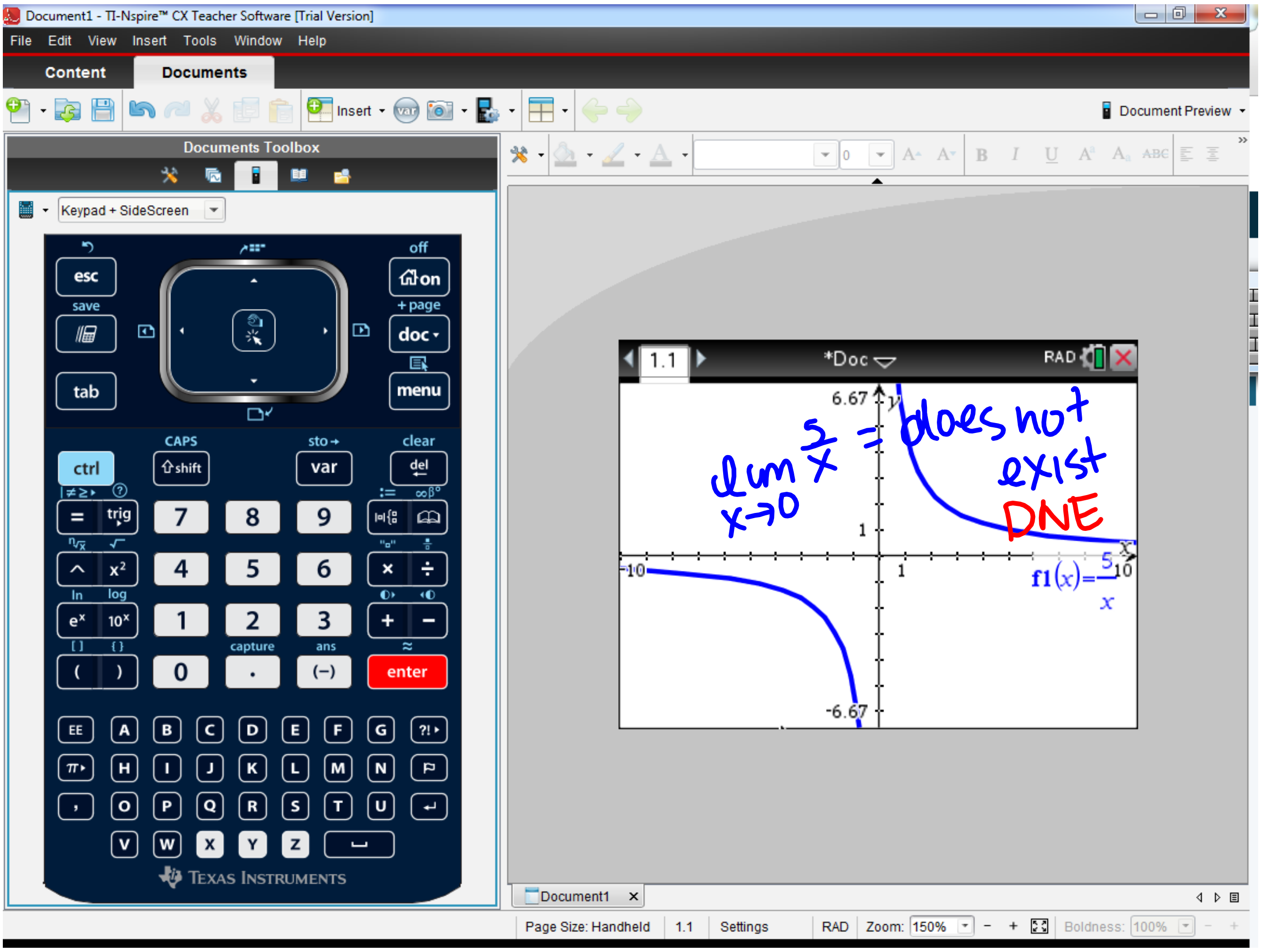
Solve graphically: The graph of $f(x) = \frac{5}{x}$ suggests that the limit does not exist.



Confirm Analytically:

We can't use substitution in this example because when x is relaced by 0, the denominator becomes 0 and the function is undefined.

This would suggest that we rely on the graph to see that the limit does not exist.


Document1 - TI-Nspire™ CX Teacher Software [Trial Version]
Documents Toolbox
Keypad + SideScreen
1.1
*Doc
RAD
6.67
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-10
10
1
lim x→0 5/x = does not exist
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f1(x)=5/x
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One-Sided and Two-Sided Limits

Sometimes the values of a function f tend to different limits as x approaches a number c from opposite sides. When this happens, we call the limit of f as x approaches c from the right the right-hand limit of f at c and the limit as x approaches c from the left the left-hand limit.

$\lim_{x \to c^+} (f(x))$

right-hand: $\lim_{x \to c^+} f(x)$ The limit of f as x approaches c from the right.

left-hand: $\lim_{x \to c^-} f(x)$ The limit of f as x approaches c from the left.

$\lim_{x \to c^-} (f(x))$

FYI...

We sometimes call $\lim_{x \to c} f(x)$ the two-sided limit of f at c to distinguish it from the one-sided right-hand and left-hand limits of f at c.

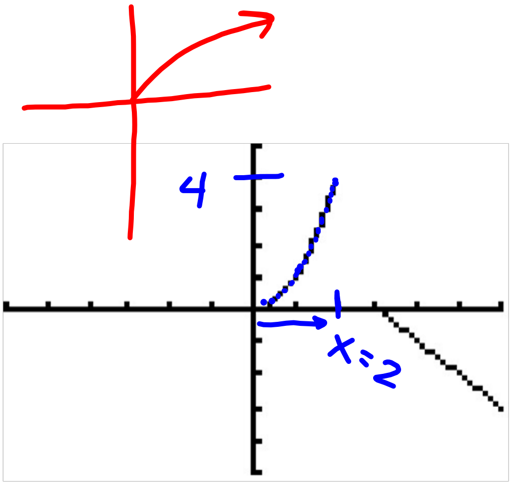
Theorem 3: One-Sided and Two-Sided Limits-

A function $f(x)$ has a limit as x approaches c if and only if the right-hand and left-hand limits at c exist and are equal. In symbols,

$$\lim_{x \to c} f(x) = L \Leftrightarrow \lim_{x \to c^+} f(x) = L \text{ and } \lim_{x \to c^-} f(x) = L.$$

You try...

Find the following limits from the given graph.



a. $\lim_{x \to 0^+} f(x) = 0$

b. $\lim_{x \to 2^+} f(x) =$ Does Not Exist

c. $\lim_{x \to 2^-} f(x) = 4$

d. $\lim_{x \to 2} f(x) =$ Does Not Exist

e. $\lim_{x \to 3^+} f(x) = 0$

Sandwich Theorem- pg 65 fig 2.7 & 2.8

If we cannot find a limit directly, we may be able to find it indirectly with the Sandwich Theorem. The theorem refers to a function f whose values are sandwiched between the values of two other functions, g and h.

If g and h have the same limit as $x \to c$ then f has that limit too.

Here's the theorem:

If $g(x) \leq f(x) \leq h(x)$ for all $x \neq c$ in some interval about c, and

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L,$$

then

$$\lim_{x\to c} f(x) = L$$

Homework

2.1 pgs.66-67 #9-48 (X3)
SKIP 12, do 14 instead